BAND SURGERIES AND CROSSING CHANGES BETWEEN FIBERED LINKS

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Abstract. We characterize cutting arcs on fiber surfaces producing fiber surfaces. As corollaries we characterize band surgeries and crossing changes between fibered links.

1. Introduction

A *fibered link* is one whose exterior fibers of S^1, so that each fiber is a Seifert surface for the link. Among the many fascinating properties of fibered links is the ability to express their exteriors as a mapping torus, thereby allowing us to encode the 3-dimensional information about the link exterior in terms of a surface automorphism. We will refer to this automorphism as the *monodromy* of the link, or of the surface. This connection yields generous amounts of information, including geometric classification (e.g. the link exterior is hyperbolic if and only if the surface automorphism is pseudo-Anosov [23]), topological information (e.g. the fiber surface is the unique minimal genus Seifert surface [2]), and methods to de-construct/re-construct fibered links.

In addition to providing beautiful examples and visualizations of link exteriors, fibered links share connections with important areas of topology, including the Berge Conjecture [14, 19, 18], as well as contact geometry due to Giroux's correspondence [8] between open books and contact structures on 3-manifolds.

In this paper we further explore constructions of fibered links in terms of the monodromy. It is known that a fiber surface of a fibered link in S^3 can be constructed from a disk by a sequence of Hopf plumbing and Hopf de-plumbing [9]. If a fiber surface has a Hopf plumbing summand, a surface obtained by cutting along the spanning arc of the Hopf annulus is another fiber surface. We will characterize all arcs on a fiber surface cutting along which gives another fiber surface. Using this result we will then study band surgeries and crossing changes between fibered links.

The paper is organized as follows: In section 2, we provide definitions and background for the tools we will use. In section 3 we study the result of cutting a fiber along an arc, and prove:

Theorem 1. *Let L be a fibered (oriented) link with fiber F, monodromy h (which is assumed to be the identity on ∂F), and suppose α is an arc in F. Let F' be the surface obtained by cutting F along α, and the resulting (oriented) link $L' = \partial F'$. The surface F' is a fiber for L' if and only if $i_{total}(\alpha) = 1$ (that is, when α is clean and alternating, or once-unclean and non-alternating).*

(See Section 2 for the definition of $i_{total}(\alpha)$.)

By [21, 12, 4] it is known that if a coherent band surgery increases the Euler characteristic of a link, then it can be isotoped onto a taut Seifert surface. Hence, such a

band surgery between fibered links corresponds to cutting the fiber surface. When a coherent band surgery changes the Euler characteristic of a link by at least two, such a band surgery is characterized by Kobayashi [15] (see Theorems 6 and 7). In section 4, we introduce a *generalized Hopf banding* and give a characterization of the remaining case.

Theorem 2. *Suppose L' is obtained from L by a coherent band surgery and $\chi(L') = \chi(L) + 1$.*

(1) *Suppose L is a fibered link. Then L' is a fibered link if and only if the fiber F for L is a generalized Hopf banding of a Seifert surface F' for L' along b.*

(2) *Suppose L' is a fibered link. Then L is a fibered link if and only if a Seifert surface F for L is a generalized Hopf banding of the fiber F' for L' along b.*

Using this result, we characterize band surgeries on torus links $T(2, p)$ or connected sums of those producing fibered links (Corollaries 2 and 4).

In section 5, we characterize when fibered links are related by crossing changes.

Theorem 3. *Suppose a link L' is obtained from a fibered link L with fiber F by a crossing change, and $\chi(L') = \chi(L)$. Then L' is a fibered link if and only if the crossing change is a Stallings twist or ε-twist along an arc α in F, where α is once-unclean and alternating with $i_\partial(\alpha) = -\varepsilon$.*

(See Section 2 for the definition of $i_\partial(\alpha)$.) In the case that $\chi(L') > \chi(L)$, such crossing changes were characterized by Scharlemann and Thompson [21] and Kobayashi [15].

As an application we can characterize the mechanisms of a system site-specific recombination. This application will be discussed in the following paper [1].

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2. Preliminaries

2.1. Surfaces.

Definition. A Seifert surface F for a link L is *taut* if it maximizes Euler characteristic over all Seifert surfaces for L. We say the Euler characteristic for the link is $\chi(L) = \chi(F)$.

Definition (see [11])**.** Let α, β be two oriented arcs properly embedded in an oriented surface F which intersect transversely. At a point $p \in \alpha \cap \beta$, define i_p to be ± 1 depending on whether the orientation of the tangent vectors $(T_p\alpha, T_p\beta)$ agrees with the orientation of F or not. If α and β intersect minimally over all isotopies fixing the boundary pointwise, then the following are well-defined:

(1) The *geometric intersection number*, $\rho(\alpha, \beta) := \sum_{p \in \alpha\cap\beta\cap int(F)} |i_p|$, is the number of intersections (without sign) between α and β in the interior of F.

(2) The *boundary intersection number*, $i_\partial(\alpha,\beta) := \frac{1}{2}\sum_{p\in\alpha\cap\beta\cap\partial F} i_p$, is half the sum of the oriented intersections at the boundaries of the arcs.
(3) The *total intersection number*, $i_{total}(\alpha,\beta) := \rho(\alpha,\beta) + |i_\partial(\alpha,\beta)|$, is the sum of the (unoriented) interior intersections between α and β, and the absolute value of half the sum of the boundary intersections between the two arcs.
(4) If F is a fiber surface with monodromy h, then we define $\rho(\alpha) := \rho(\alpha, h(\alpha))$, and $i_\partial(\alpha) := i_\partial(\alpha, h(\alpha))$, and $i_{total}(\alpha) := i_{total}(\alpha, h(\alpha))$.

2.2. **Sutured manifolds.**

Definition (see [5, 20, 10])**.** A *sutured manifold*, (M,γ), is a compact 3-manifold M, with a set $\gamma \subset \partial M$ of mutually disjoint annuli, $A(\gamma)$, and tori, $T(\gamma)$, satisfying the orientation conditions below. (We will only consider the case when $T(\gamma) = \emptyset$.) Call the core curves of the annuli $A(\gamma)$ the *sutures*, and denote them $s(\gamma)$. Let $R(\gamma) = \partial M \setminus int(A(\gamma))$.

(1) Every component of $R(\gamma)$ is oriented, and $R_+(\gamma)$ (respectively, $R_-(\gamma)$) denotes the union of the components whose normal vectors point out of (resp., into) M.
(2) The orientations of $R_\pm(\gamma)$ are consistent with the orientations of $s(\gamma)$.

We will often simplify notation and write $(M, s(\gamma))$ in place of (M,γ).

Definition (see [5, 20, 10])**.** We say that a sutured manifold (M,γ) is a *trivial sutured manifold* if it is homeomorphic to $(F \times I, \partial F \times I)$, for some compact, bounded surface F, with $R_+(\gamma) = F \times \{1\}$, $R_-(\gamma) = F \times \{0\}$, and $A(\gamma) = \partial F \times I$.

Definition (see [5, 20, 10])**.** Suppose F is a Seifert surface for an oriented link L. Then $(n(F), L) = (F \times I, \partial F \times \{\frac{1}{2}\})$ is a trivial sutured manifold. We call $(M \setminus n(F), L)$ the *complimentary sutured manifold.*

Definition (see [5, 20, 10])**.** A properly embedded disk D in (M,γ) is a *product disk* if $\partial D \cap A(\gamma)$ consists of two essential arcs in $A(\gamma)$. A *product decomposition* is an operation to obtain a new sutured manifold (M',γ') from a sutured manifold (M,γ) by decomposing along an oriented product disk (see [5]). We denote this

$$(M,\gamma) \overset{D}{\rightsquigarrow} (M',\gamma').$$

Definition (see [10])**.** A properly embedded surface P in a compression body N is called *boundary compressible toward* $\partial_+ N$ if there exists a disk D in N such that $D \cap P = \eta$ (an arc in ∂D) and $D \cap \partial_+ N = \beta$ (an arc in ∂D), with $\eta \cap \beta = \partial\eta = \partial\beta$, $\eta \cup \beta = \partial D$, and either η is essential in P or η is inessential in P and the boundaries of all disk components of $cl(P \setminus \eta)$ intersect $\partial(\partial_- N) \times I$ (i.e. the sutures).

Theorem 4 ([10])**.** *Assume that $\partial P \cap \partial_+ W \neq \emptyset$ and P is incompressible and boundary incompressible toward $\partial_+ W$. Then P is either*

(1) an annulus such that one boundary component is contained in $\partial_+ W$ and the other is contained in $\partial_- W$. (This is called a product annulus.*)*
(2) a disk whose boundary component is contained in $\partial_+ W$, or
(3) a product disk in W.

Conventions. *Let us establish some informal conventions to aid in visualization:*

(1) $F \times [0,1]$ *will refer to a product where the* $[0,1]$ *component is 'vertical,' with* $F \times \{0\}$ *on the 'top,' and* $F \times \{1\}$ *on the 'bottom'.*
(2) *Correspondingly, the orientation of* F *will be such that* $F \times \{0\}$ *corresponds to the 'positive' side of* F.
(3) $[0,1] \times F$ *will refer to a product where the* $[0,1]$ *component is 'horizontal,' with* $\{0\} \times F$ *on the 'left,' and* $\{1\} \times F$ *on the 'right'.*
(4) *A fibered link complement will be thought of as coming from a mapping torus* $F \times [0,1]/h$, *where* $h : F \times \{1\} \to F \times \{0\}$, *so that the product disk determined by* α *and* $h(\alpha)$ *will emanate 'downwards' from* α *in* $F \times \{1\}$, *and 'upwards' from* $h(\alpha)$ *in* $F \times \{0\}$.

3. Cutting arcs in fiber surfaces

Let L be a fibered (oriented) link in a manifold M with fiber F, monodromy h (which is assumed to be the identity on ∂F), and suppose α is an arc in F. Assume α and $h(\alpha)$ have been isotoped in F, fixing the endpoints, so intersect minimally. If the endpoints of $h(\alpha)$ emanate to opposite sides of α, then $|i_\partial(\alpha)| = 1$. In this case, α is known as *alternating.* Otherwise, $|i_\partial(\alpha)| = 0$, and α is called *non-alternating.* If $\rho(\alpha) = 0$, then α is said to be *clean.* If $\rho(\alpha) = n > 0$, α is said to be *n-unclean.*

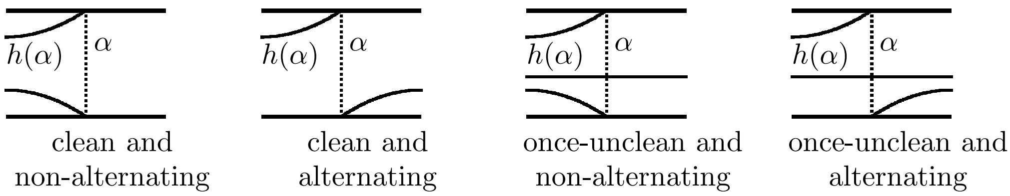


Figure 1

Let F' be the surface obtained by cutting F along α, and the resulting (oriented) link $L' = \partial F'$. We now restate Theorem 1.

Thereom 1. *The surface* F' *is a fiber for* L' *if and only if* $i_{total}(\alpha) = 1$ *(that is, when* α *is clean and alternating, or once-unclean and non-alternating).*

Consider the fiber F, and a small product neighborhood $n(F) = F \times I$. This is a trivial sutured manifold, $(n(F), \partial F)$. Let D_- be the product disk $\alpha \times I$. Now, because F is a fiber for L, the complementary sutured manifold $(M \smallsetminus n(F), \partial F)$ is also trivial. Let D_+ be the product disk determined by $\alpha \subset F \times \{1\}$ and $h(\alpha) \subset F \times \{0\}$, properly embedded in $M \smallsetminus n(F)$. As D_- is a product disk for $(n(F), \partial F)$, we may decompose along this disk to get another trivial sutured manifold, namely $(n(F'), \partial F')$.

Recall that the manifold $n(F')$ was obtained by removing a small product neighborhood of D_-, say $[0,1] \times D_- = [0,1] \times (\alpha \times [0,1])$, from $n(F)$. Let B be the ball $[-1,2] \times (\alpha \times [-1,1])$. Now, attach to $(n(F'), \partial F')$ the 1-handle $([-1,2] \times (\alpha \times [-1,0]))$, (attached along $([-1,0] \times (\alpha \times \{0\}))$ and $([1,2] \times (\alpha \times \{0\})))$. Call the resulting sutured manifold $(N_1, \partial F')$ (see Figure 2). We will refer to $(F' \times \{1\}) \subset \partial N_1$ as $\partial_- N_1$, and $\partial N_1 \smallsetminus ((F' \times \{1\}) \cup (\partial F' \times I))$ as $\partial_+ N_1$.

Now, we can modify D_+ to a new disk D'_+ in $M \smallsetminus N_1$ as follows:

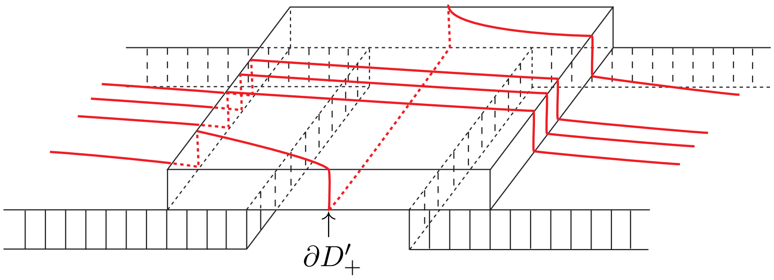


FIGURE 2. The sutured manifold $(N_1, \partial F')$. The boundary of the modified disk $\partial D'_+$ reflects the pattern of the product disk D_+.

(1) Let $\widetilde{D_+} = D_+ \setminus B$. Note that $h(\alpha)$ corresponds through vertical projection in B to arcs in $(\{-1\} \times (\alpha \times [-1, 0])) \cup ([-1, 2] \times (\alpha \times \{-1\})) \cup (\{2\} \times (\alpha \times [-1, 0]))$.
(2) Extend the sub-arc $\alpha \times \{1\}$ of $\widetilde{D_+}$ through vertical projection in B to $\alpha \times \{0\}$.

Lemma 1. *The co-core of the* 1*-handle is the unique compressing disk in* N_1 *disjoint from the sutures. Furthermore, every product disk in* $(N_1, \partial F')$ *can be made disjoint from this disk.*

Proof. Let D be the co-core of the 1-handle, and suppose D' is either (A) a distinct compressing disk disjoint from $\partial F'$ or (B) a product disk for the sutured manifold; in either case chosen so as to minimize $|D \cap D'|$. Suppose l were a loop of intersection innermost in D. Then l bounds a subdisk $\widetilde{D}$ of D and a subdisk $\widetilde{D'}$ of D'. These two disks co-bound a sphere, which then bounds a 3-ball because N_1 is irreducible. This sphere provides a means of isotoping $\widetilde{D'}$ to $\widetilde{D}$, which reduces the number of loops in $D \cap D'$, contradicting minimality. Thus, we may suppose that $D \cap D'$ consists only of arcs. In this case, an outermost arc in D provides a boundary compression of D', separating D' into two disks. At least one of these disks is a disk of the same type as D' (i.e., (A) or (B) above), but which intersects D fewer times than D', again contradicting minimality.

So we may assume now that $D' \cap D = \emptyset$. Hence, we may isotope D' completely off of the 1-handle. In case (B), this establishes the second statement. In case (A), D' is a compressing disk for $F' \times \{0\}$ in $F' \times I$, an impossibility. This contradiction establishes the first statement. □

Now, we attach a 2-handle to N_1 along a neighborhood of $\partial D'_+$. Call the resulting sutured manifold $(N_2, \partial F')$, and keep track of the attaching annulus, $A = n(\partial D'_+)$, on the one hand thought of as contained in the boundary of N_1, and on the other hand considered to be properly embedded in N_2. Observe that since D_+ was a product disk for the trivial sutured manifold $(M \setminus n(F), \partial F)$ which is homeomorphic to N_1, attaching the 2-handle along $\partial D'_+$ results in the same manifold as decomposing $(M \setminus n(F), \partial F)$ along D_+. Furthermore, the sutures in $(M \setminus N_2, \partial F')$ can be slid along D'_+, and can be seen to agree with the result of this decomposition. Therefore, as a sutured manifold, $(M \setminus N_2, \partial F')$ is the same as the result of decomposing $(M \setminus n(F), \partial F)$ along the product disk D_+, and is thus a trivial sutured manifold.

We conclude then that F' is a fiber in a fibration for L' if and only if $(N_2, \partial F')$ is trivial.

Remark 1. We remind the reader that $(N_1, \partial F')$ is simply constructed from $(n(F'), \partial F')$ by attaching a 1-handle along $F' \times \{0\}$, and that $(N_2, \partial F')$ is constructed by attaching a 2-handle to $(N_1, \partial F')$.

Lemma 2. *If $(N_2, \partial F')$ is trivial, then there exists a compressing disk for $(N_1, \partial F')$, disjoint from the sutures, which intersects the boundary of D'_+ exactly once.*

Proof. If $(N_2, \partial F')$ is trivial, then there exists a full product decomposition,

$$(N_2, \partial F') \overset{D_1}{\rightsquigarrow} (N_2^1, \gamma_2^1) \overset{D_2}{\rightsquigarrow} (N_2^2, \gamma_2^2) \overset{D_3}{\rightsquigarrow} \cdots \overset{D_k}{\rightsquigarrow} (D^2, \partial D^2).$$

Suppose that there exists a product disk $D := D_i$ which intersects the 2-handle. Then consider intersections between D and the attaching annulus A.

Any loops of intersection that are trivial in A can be removed by an innermost disk argument: compress D along the sub-disk cut off from A. One of the resulting components is a disk with the same boundary, intersecting the 2-handle fewer times. Similarly, any arcs that are trivial in A can be removed by an outermost arc argument: boundary compress D along the sub-disk cut off from A. One of the resulting components is a product disk for N_2, intersecting the 2-handle fewer times.

So we will assume that all intersections between D and A are essential in A. Observe, then, that this implies all intersections are loops, or all intersections are arcs, since disjoint essential loops and arcs in an annulus do not exist.

Suppose there are arcs of intersection. Since A is contained in $\partial_+ N_1$, all such arcs have both endpoints on $\partial_+ N_1$. Consider an outermost such arc, δ, cutting off a subdisk $\widehat{D}$. Then, since δ is essential in A, the 'outer' arc $\partial\widehat{D} \setminus \delta$ must be essential in $\partial_+ N_1 \setminus A$. This is because its endpoints are incident to the two boundary components of A, whose core is an essential curve in ∂N_1. Then $\widehat{D}$ is a compressing disk for N_1, disjoint from the sutures, and $\partial\widehat{D}$ consists of an arc in the complement of $\partial D'_+$, and an essential arc in A, which then intersects $\partial D'_+$ exactly once. Thus, $\widehat{D}$ is the desired disk.

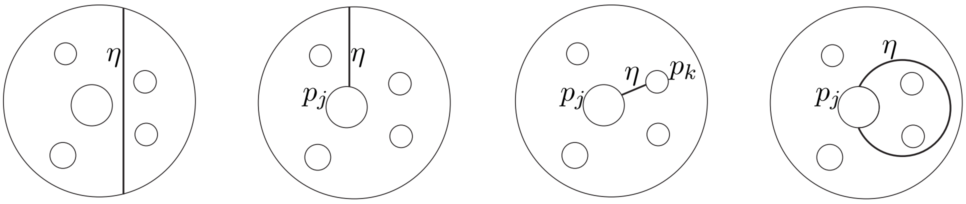
Suppose instead that all intersections between A and D are loops essential in A. Consider the component of $D \cap N_1$ which retains the outermost boundary, crossing the sutures of N_2. Call this punctured product disk $\widehat{D}$, and note that all punctures, p_j, are cores of A. By Theorem 4 of Goda, $\widehat{D}$ is either compressible in N_1, or boundary compressible towards $\partial_+ N_1$. We will show that $\widehat{D}$ can be compressed and boundary compressed completely to a product disk for $(N_1, \partial F')$ which is disjoint from $\partial D'_+$. We introduce these definitions formally.

Definition. A *punctured product disk* is a planar surface properly embedded in a sutured manifold (M, γ), such that one boundary component, called the *outer boundary*, crosses γ exactly twice, and the rest of the boundary components, called *punctures*, are disjoint from γ. We say that a punctured product disk is *anchored* to a curve c in ∂M if all the punctures are parallel to c in ∂M. We will also consider a product disk to be a punctured product disk (with zero punctures), and say it is anchored to c if the boundary of the disk is disjoint from c.

We observe that a punctured product disk in $(N_1, \partial F')$ anchored to the core of A can be extended to a product disk in $(N_2, \partial F')$ just by capping off each of the puncture components with a disk parallel to the core of the 2-handle in N_2.

A boundary compressing disk, Δ, consists of two arcs: η contained in $\widehat{D}$, and β contained in $\partial_+ N_1$. We will find that there are six possible types of boundary compressing disks for a punctured product disk anchored to the curve c:

Type 1. The arc η connects two points of the outer boundary of $\widehat{D}$.
Type 2. The arc η connects the outer boundary of $\widehat{D}$ and a puncture p_j.
Type 3. The arc η connects two distinct punctures, p_j and p_k, both parallel to the core of A, with β contained in A.
Type 3'. The arc η connects two distinct punctures, p_j and p_k, and either β is not contained in A, or p_j and p_k are not both parallel to the core of A.
Type 4. The arc η connects two points of one puncture, p_j, and β is trivial in $(\partial_+ N_1)|(\partial \widehat{D})$.
Type 4'. The arc η connects two points of one puncture, p_j, and β is essential in $(\partial_+ N_1)|(\partial \widehat{D})$.



Type 1 Type 2 Type 3 (Type 3') Type 4 (Type 4')

FIGURE 3

Only one of these types of boundary compressions will take us out of the class of punctured product disks anchored to the core of A, namely a Type 3' boundary compression. However, there is a broader class of surfaces that will be preserved through any boundary compression towards $\partial_+ N_1$ that we wish to perform. So first, let us observe what happens when we perform a boundary compression towards $\partial_+ N_1$ of Type 3': the result has a new puncture which is no longer parallel to a core of A. The new puncture, instead, runs around an original puncture, along β, around another original puncture, and then back along β. In this case, all the punctures of the new punctured product disk lie inside the punctured torus $A \cup n(\beta)$. Call this punctured torus T.

Definition. We will say that a punctured product disk is *almost anchored* to the core of A if all of the punctures are either parallel to the core of A, or to ∂T.

We observe a very important feature of T, namely that ∂T is an essential curve in $\partial_+ N_1$, but a trivial curve in $\partial_+ N_2$. This is because the core of A is essential in $\partial_+ N_1$, whereas the two boundary components of A bound disks in $\partial_+ N_2$ (the frontier disks of the 2-handle), allowing the curve ∂T to be isotoped down to a point through $\partial_+ N_2$. A direct consequence of this observation is the following lemma.

Claim 1. *A punctured product disk in $(N_1, \partial F')$ almost anchored to the core of A can be extended to a product disk in $(N_2, \partial F')$.*

Proof. Every curve parallel to the core of A can be capped off in N_2 with disk parallel to the core of the 2-handle, and every curve prallel to ∂T can be capped off with a boundary parallel disk in N_2. □

Now, we consider the six possible types of boundary compressing disks towards $\partial_+ N_1$ for $\widehat{D}$, a punctured product disk *almost* anchored to the curve c:

Type 1. The arc η connects two points of the outer boundary of $\widehat{D}$.
Type 2. The arc η connects the outer boundary of $\widehat{D}$ and a puncture p_j.
Type 3. The arc η connects two distinct punctures, p_j and p_k, both parallel in $\partial_+ N_1$, with β contained in the annulus of parallelism.
Type 3'. The arc η connects two distinct punctures, p_j and p_k, and β is not contained in an annulus between them.
Type 4. The arc η connects two points of one puncture, p_j, and β is trivial in $(\partial_+ N_1)|(\partial\widehat{D})$.
Type 4'. The arc η connects two points of one puncture, p_j, and β is essential in $(\partial_+ N_1)|(\partial\widehat{D})$.

We will show that any sequence of boundary compressions can be chosen so as to preserve an almost anchored punctured product disk.

Claim 2. *If $\widehat{D}$ is a punctured product disk in $(N_1, \partial F')$ almost anchored to the core of A, and Δ is a boundary compression towards $\partial_+ N_1$ of Type 1 or 2, then the result of the boundary compression has a component which is a punctured product disk in $(N_1, \partial F')$ almost anchored to the core of A, having fewer punctures than $\widehat{D}$.*

Proof. Suppose that Δ is a boundary compression of Type 1. Then η is separating in $\widehat{D}$, and either has punctures on both sides, or all the punctures opposite the side which has boundary intersecting $\partial_- N_1$. Either way, one of the components resulting from the boundary compression is still a punctured product disk in $(N_1, \partial F')$, with a (strict) subset of the punctures of $\widehat{D}$.

Suppose instead that Δ is a boundary compression of Type 2. Then the result of boundary compressing still has outer boundary crossing the sutures exactly twice, and has all the punctures of $\widehat{D}$ except p_j. □

Claim 3. *If $\widehat{D}$ is a punctured product disk in $(N_1, \partial F')$ almost anchored to the core of A, and Δ is a boundary compression towards $\partial_+ N_1$ of Type 3, then the result of the boundary compression has a puncture which is trivial in $\partial_+ N_1$. Capping off this boundary component with a disk yields a punctured product disk in $(N_1, \partial F')$ almost anchored to the core of A, having fewer punctures than $\widehat{D}$.*

Proof. The two punctures connected by β are parellel in $\partial_+ N_1$, and β lies in the anuulus of their parallelism. Hence, the result of boundary compressing along Δ will have a new boundary component which traverses the puncture p_j and then p_k in the opposite direction. This loop will be trivial in $\partial_+ N_1$. The outer boundary and the other punctures remain unchanged, so capping off this new trivial boundary component returns a new punctured product disk in $(N_1, \partial F')$ with fewer punctures than $\widehat{D}$. □

Claim 4. *If $\widehat{D}$ is a punctured product disk in $(N_1, \partial F')$ almost anchored to the core of A, and Δ is a boundary compression towards $\partial_+ N_1$ of Type 3', then the result of the*

boundary compression is a punctured product disk in $(N_1, \partial F')$ almost anchored to the core of A, having fewer punctures than $\widehat{D}$.

Proof. The arc β connects two different punctures p_j and p_k. As ∂T is separating in $\partial_+ N_1$, and Δ is of Type 3', either both are parallel to the core of A and β is not contained in A, or p_k is parallel to ∂T and p_j is parallel to the core of A. In either case, β must lie inside T.

In the first case, we observe that performing the boundary compression replaces p_j and p_k with the band sum of p_j and p_k along the arc β. Then, cutting the punctured torus T along p_j and p_k would result in a thrice-punctured sphere. Up to isotopy, there is a unique arc in a thrice-punctured sphere connecting two fixed boundary components, and the band sum of those two boundary components along such an arc is the third boundary component of the punctured sphere. Therefore, the result of boundary compressing $\widehat{D}$ along Δ replaces the two punctures p_j and p_k with a single puncture which is parallel to ∂T. As the other punctures and the outer boundary are unaffected, the result is a punctured product disk in $(N_1, \partial F')$ almost anchored to the core of A.

In the second case, we have a similar phenomenon. Recall that p_j is parallel to the core of A and p_k is parallel to ∂T. Then, cutting the punctured torus T along p_j and p_k would again result in a thrice-punctured sphere. Up to isotopy, there is again a unique arc connecting the two fixed boundary components, and the band sum of those two boundary components along such an arc is the third boundary component of the punctured sphere. The third boundary component of $T|(p_j \cup p_k)$ is a copy of p_j. Therefore, the result of boundary compressing $\widehat{D}$ along Δ replaces the two punctures p_j and p_k with a single puncture which is parallel to the core of A. As the other punctures and the outer boundary are unaffected, the result is a punctured product disk in $(N_1, \partial F')$ almost anchored to the core of A. □

Claim 5. *If $\widehat{D}$ is a punctured product disk in $(N_1, \partial F')$ almost anchored to the core of A, and Δ is a boundary compression towards $\partial_+ N_1$ of Type 4, then $\widehat{D}$ is compressible in N_1, and there exists a component after the compression, $\widehat{D}'$, so that $\widehat{D}' \cap N_1$ is a punctured product disk in $(N_1, \partial F')$ almost anchored to the core of A, having fewer intersections with A than $\widehat{D}$.*

Proof. As Δ is a boundary compression of Type 4, β is trivial in $\partial_+ N_1 \setminus \partial \widehat{D}$ and cuts off a disk, D_c, from $\partial_+ N_1 \setminus \partial \widehat{D}$. So, D_c guides an istopy of $\partial \Delta$ to $int(\widehat{D})$. Now, replacing the interior of the loop $\partial \Delta$ in $\widehat{D}$ with Δ itself, we produce a new punctured product disk almost anchored to the core of A. Since the interior of Δ was contained in N_1, and η necessarily separated $\widehat{D}$, the new punctured product disk intersects A fewer times. □

We will treat boundary compressions of Type 4' slightly differently, depending on the type of puncture involved. If the puncture is parallel to the core of A, then whenever we see a boundary compression of Type 4', we can perform an actual compression of the punctured product disk in N_2 in a way which still allows us to simplify the result. On the other hand, if the puncture is parallel to ∂T, we will need to show extra care to ensure that we do not end up with a punctured product disk that intersects the core of

A. In this case, we will either be able to perform the boundary compression, or we will show that there is a better boundary compression we can perform instead.

Claim 6. *If $\widehat{D}$ is a punctured product disk in $(N_1, \partial F')$ almost anchored to the core of A, and Δ is a boundary compression towards $\partial_+ N_1$ of Type 4', incident to a puncture parallel to the core of A, then $\widehat{D}$ is compressible in N_2 via a compressing disk which intersects the 2-handle in exactly one disk. The result of the compression has a component whose intersection with N_1 is a punctured product disk almost anchored to the core of A, and either*

- *has fewer punctures than $\widehat{D}$, or*
- *has the same number of punctures as $\widehat{D}$, and admits a boundary compression towards $\partial_+ N_1$ of Type 3'.*

Proof. The arc β is incident to a single puncture p of $\widehat{D}$ at two points. These points divide the puncture into two sub-arcs, say p' and p'', and since the puncture is parallel to the core of A, β intersects ∂A in two points on the same component, dividing this loop into two sub-arcs, say a' and a''. So there are two rectangles: $R' \subset \partial_+ N_1$ bounded by $\beta \cap A$ on two sides, a' on one side, and p' on one side; and a similar rectangle R'' bounded by $\beta \cap A$, a'', and p''. Push the interior of these rectangles slightly into the interior of N_1.

Consider the two annuli $A'_c = \Delta \cup R'$, and $A''_c = \Delta \cup R''$. Both have one boundary component in $\widehat{D}$ (namely, $\eta \cup p'$ or $\eta \cup p''$), and one component in ∂N_1 (namely, $(\beta \setminus A) \cup a'$ or $(\beta \setminus A) \cup a''$). As $\widehat{D}$ can be extended to a product disk in $(N_2, \partial F')$, the loops in $\widehat{D}$ both cut off a sub-disk in N_2. Capping off either annulus with the appropriate disk results in a disk, D'_p or D''_p, in N_2, whose boundary lies in $\partial_+ N_1 \cap \partial_+ N_2$.

However, since Δ is a Type 4' boundary compression, and β is incident to puncture parallel to the core of A, β is essential in $\partial_+ N_1 \setminus A$. So β is either non-separating in $\partial_+ N_1 \setminus A$, or it is separating and neither component is a disk. It cannot be the case that the loop $(\beta \setminus A) \cup a'$ is essential in $\partial_+ N_2$, for then $A'_c \cup D'_p$ would be a compression disk for $\partial_+ N_2$ in N_2, and $(N_2, \partial F')$ was assumed to be a trivial sutured manifold. Therefore, β must have been separating, and some component of $(\partial_+ N_1 \setminus \partial \widehat{D})|\beta$, while not a disk, must become a disk after the addition of the 2-handle to N_1. Hence, one of these components must be an annulus which is capped off to become a disk by the addition of the frontier disks of the 2-handle. In other words, β runs along some arc ζ in $\partial_+ N_1$ joining two punctures, around the second puncture, and then back back along ζ to the first puncture (see Figure 4).

Now, both D'_p and D''_p are disks in N_2, with boundaries trivial in $\partial_+ N_2$ and differing exactly by the arc p' or p''. By choosing the appropriate arc, say p', we can assume $\partial D'_p$ separates the two frontier disks of the 2-handle. Since $\partial_+ N_2$ is incompressible, the disk D'_p must be boundary parallel, so let D'_∂ be a slight push-off of the disk in $\partial_+ N_2$ bounded by $\partial D'_p$. Observe that since $\partial D'_p$ separates the two frontier disks of the 2-handle, D'_∂ intersects the 2-handle exactly once.

We can now compress $\widehat{D}$ along a slight push-off of the disk $A'_c \cup D'_\partial$ in N_2. One resulting component contains the outer boundary of $\widehat{D}$, and let $\widehat{D}'$ be the intersection

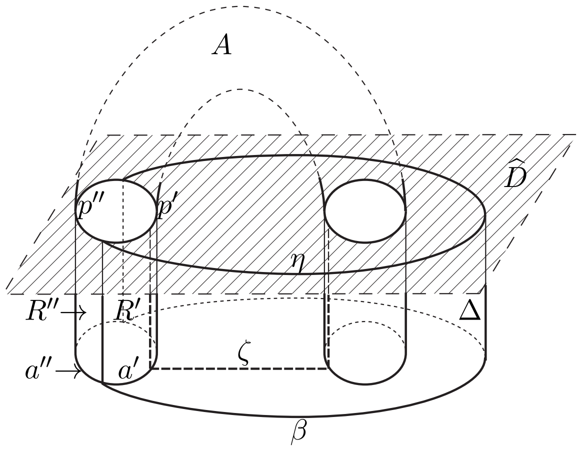


FIGURE 4. Boundary compression of type 4'.

of that component with N_1. Then $\widehat{D}'$ is a punctured product disk almost anchored to the core of A.

If the loop $\eta \cup p'$ contained more than one puncture in $\widehat{D}$, then $\widehat{D}'$ will possess strictly fewer punctures than $\widehat{D}$ did. Otherwise, since D'_∂ is boundary parallel in N_2, a thickened neighborhood of the arc ζ can be chosen to intersect $\widehat{D}'$ in a neighborhood of an arc running from p' to the new puncture along $A'_c \cup D'_\partial$, and to intersect $\partial_+ N_1$ in a neighborhood of an arc also running from p' to the new puncture along a sub-arc of ζ. In this neighborhood of ζ, then, we see a Type 3' boundary compressing disk for $\widehat{D}'$ whose β-arc connects p to the new (single) puncture (see Figure 4). $\square$

Claim 7. *If $\widehat{D}$ is a punctured product disk in $(N_1, \partial F')$ almost anchored to the core of A, and Δ is a boundary compression towards $\partial_+ N_1$ of Type 4', incident to a puncture which is parallel to ∂T, and so that the β-arc of Δ does not intersect the core of A, then the result of the boundary compression has a component which is a punctured product disk almost anchored to the core of A, having fewer punctures than $\widehat{D}$.*

Proof. We first claim that β must lie inside the punctured torus T. Suppose, instead, that β lies outside of T.

Then the arc β is incident to a single puncture of $\widehat{D}$ parallel to ∂T at two points. These points divide the puncture into two sub-arcs, say p' and p''. We may slide Δ so that β never intersects A, and one of the sub-arcs, say p' is also disjoint from A. Then let R' be the rectangular boundary of a neighborhood of the arc p', with opposite sides in Δ, one side in $\widehat{D}$, and one side in $\partial_+ N_1$. Taking the union of Δ with R', and removing the two corners cut off from Δ by R', we construct an annulus A' with one boundary component arising from $\beta \cup p'$ in $\partial_+ N_1$, and one boundary component arising from $\eta \cup' p$ in $\widehat{D}$.

Now, β is essential in $(\partial_+ N_1)|(\partial \widehat{D}) \subset \partial_+ N_2$. The puncture $p' \cup p''$ is separating in $\partial_+ N_2$, and in light of Claim 1, is trivial in $\partial_+ N_2$, so $\beta \cup p'$ is an essential loop in $\partial_+ N_2$.

But Claim 1 also implies $\eta \cup p'$ bounds a disk in N_2. Hence, capping off A' with this disk would provide a compressing disk for $\partial_+ N_2$, which contradicts the triviality of N_2.

So we may assume that β is inside the punctured torus T. Then β is incident to a single puncture p_j of $\widehat{D}$. Now, it suffices to observe that there is a unique arc in the thrice-punctured sphere $T|$(the core of A) with both endpoints on a fixed boundary component, and the band sum of such a boundary component to itself along this arc results in two curves, each parallel to one of the remaining two boundary components. Thus, the resulting curves are both parallel to the core of A, and the result of the boundary compression has one component which is a punctured product disk almost anchored to the core of A, and having fewer punctures than $\widehat{D}$. □

There is an additional assumption in Claim 7, namely that β does not intersect the core of A. As long as there exist some punctures of $\widehat{D}$ which are parallel to the core of A, β never will intersect the core of A. However, it is possible that a sequence of boundary compressions results in a punctured product disk, almost anchored to the core of A, with every puncture parallel to ∂T, so that there exists a boundary compression for the punctured product disk of Type 4', whose β arc does intersect the core of A. Such a boundary compression would be dire for our project, as subsequent boundary compressions could end with a product disk for $(N_1, \partial F')$ which intersects the core of A.

Accordingly, we will offer the following definition:

Definition. A Type 4' boundary compression for a punctured product disk almost anchored to the core of A will be called *dire* if β intersects the core of A.

We will now turn our attention to showing that only certain types of dire Type 4' boundary compressions can exist, and that these can be avoided.

Claim 8. *If $\widehat{D}$ is a punctured product disk in $(N_1, \partial F')$ almost anchored to the core of A, and Δ is a dire Type 4' boundary compression, then there exists a different punctured product disk in $(N_1, \partial F')$ almost anchored to the core of A, having fewer punctures than $\widehat{D}$, which either has a boundary compression towards $\partial_+ N_1$ which is not dire, or is an unpunctured product disk anchored to the core of A.*

Proof. First, we claim that the β-arc of a dire Type 4' boundary compression does not intersect the core of A more than once. Suppose to the contrary, and we will find a punctured lens space in N_2.

The curve ∂T bounds a disk D_∂ in the $\partial_+ N_2$. We take a collar $D_\partial \times [0,1]$ of D_∂ in N_2, where $D_\partial \times \{0\} \subset \partial_+ N_2$, and $D_\partial \times (0,1]$ is contained in the interior of N_2. Then a union of $D_\partial \times [\frac{1}{2}, 1]$ and ((the 2-handle) $\setminus (D_\partial \times [0,1))$) is solid torus V (see Figure 5).

Now, move the punctured product disk $\widehat{D}$ from $D_\partial \times \{0\}$ to $D_\partial \times \{1\}$ along the collar, together with the boundary compression disk Δ. The arc η in $\widehat{D}$ separates $\widehat{D}$ into two punctured disks. Let D_η be the component without the outer boundary of $\widehat{D}$.

Then, $\Delta \cup D_\eta$ can be extended to a properly embedded disk in $N_2 \setminus V$, since each puncture bounds a disk in N_2 passing through $D_\partial \times (0, \frac{1}{2})$. Hence, there exists a punctured lens space, obtained by slightly thickening the disk $\Delta \cup D_\eta$, and attaching it to V along β and p'. It is impossible that a punctured lens space appear in a surface cross an interval, $N_2 \cong F' \times [0,1]$.

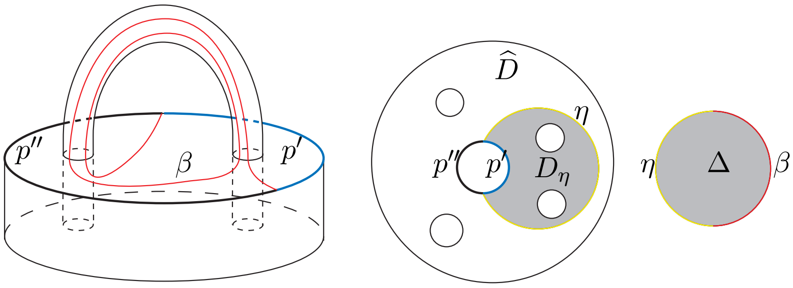


FIGURE 5. The solid torus V, and the attaching curve of a 2-handle which would define a punctured lens space in N_2.

Thus, we may assume that the β-arc of the dire boundary compression disk Δ intersects the core of A exactly once. Suppose that β has its endpoints adjacent to the puncture p.

From the definition of dire, we know that p is parallel to ∂T. But the only way that curves parallel to ∂T arise is from a Type 3' boundary compression. Hence, the puncture p arose as the result of some Type 3' boundary compression Δ' incident to two punctures, p_1 and p_2, both parallel to the core of A. Notice that since Δ intersects c just once, we can isotope the endpoints of the β-arc around p so that β lies inside the annulus defined by p_1 and p_2.

Now, let T' be the sub-(punctured torus) of T bounded by p. Observe that the boundary compression disk Δ' is disjoint from any other boundary compression disk whose β-arc lies within T_i, so that the Δ' boundary compression actually commutes with any such boundary compression.

So, consider the sequence of boundary compressions starting with Δ', and ending with Δ. The puncture p persists through these boundary compressions, so any occuring outside of T' must not involve p, and so the Δ' boundary compression commutes with these as well. Thus, we may perform all of the boundary compressions in the sequence, excepting Δ and Δ'. The result is a punctured product disk which still has p_1 and p_2 as punctures. Since the β-arc of Δ lies in the annulus defined by p_1 and p_2, and all the other punctures parallel to the core of A have been eliminated, Δ is a boundary compressing disk for this punctured product disk. Now, however, Δ corresponds is a Type 3 boundary compression, and is not dire. Performing the Type 3 boundary compression defined by Δ and then capping off the resulting trivial puncture results in a punctured product disk almost anchored to the core of A which has one fewer puncture than $\widehat{D}$, as a Type 3' boundary compression was replaced with a Type 3 boundary compression.

Hence, we can replace the entire sequence of boundary compressions leading up to the dire Type 4' boundary compression with a new sequence which results in a different punctured product disk (almost anchored to c), for which the dire boundary compression does not exist. While a new dire boundary compression may arise, a repeated application of this argument must eventually produce a punctured product disk with a non-dire boundary compression, or an honest product disk. $\square$

We can now finish the proof of Lemma 2. By Theorem 4, $\widehat{D}$ is either compressible, or boundary compressible. By Claims 1 through 8, the result is disjoint from ∂D_+, and will again be compressible or boundary compressible by Theorem 4. Hence, we can compress or boundary compress $\widehat{D}$ completely to produce a product disk for $(N_2, \partial F')$ which is disjoint from $\partial D'_+$. Decomposing along such a disk preserves being a trivial sutured manifold, so we can take a new product disk and repeat the process. Suppose, then, that there exists a full product decomposition for $(N_2, \partial F')$ in which every product disk D is disjoint from the 2-handle.

In this case, the result of the decomposition is a trivially sutured 3-ball in which the suture is disjoint from A. We can then decompose $(N_1, \partial F')$ along all of these product disks before attaching the 2-handle, obtaining

$$(N_1, \partial F') \overset{D_1}{\rightsquigarrow} (N_1^1, \gamma_1^1) \overset{D_2}{\rightsquigarrow} \cdots \overset{D_k}{\rightsquigarrow} (N_1^k, \gamma_1^k).$$

So (N_1^k, γ_1^k) is a sutured manifold with the property that there is a 2-handle attachment disjoint from γ_1^k which produces a trivially sutured 3-ball.

However, by Lemma 1, each of the disks D may be taken to be disjoint from the 1-handle as well. We conclude that N_1^k is a disk times an interval plus the 1-handle. Hence, N_1^k is a solid torus. In this case, the existence of the 2-handle ensures that it is an unknotted solid torus, and thus that $\partial D'$ intersects ∂D in exactly one point, as claimed. □

Proof of Theorem 1. Combining the results of Lemmas 1 and 2, we see that if F' is a fiber for L', then $|\partial D \cap \partial D'_+| = 1$. Since $\partial D'_+$ reflects the product disk D_+, and therefore the pattern of α and $h(\alpha)$ on F, this shows that α must be either alternating and clean, or non-alternating and once-unclean.

Conversely, we know that if α were alternating and clean, then F' would be the fiber of a fibration for L'. Thus, it remains to show that if α is non-alternating and once-unclean, then $(N_2, \partial F')$ is trivial. This is shown by observing that in this case, ∂D and $\partial D'_+$ form a canceling pair. The sutured manifold $(N_2, \partial F')$ is the result of attaching the 1-handle with co-core D to $(F' \times I, \partial F')$, and then the 2-handle along $\partial D'_+$. As these are canceling handles, this is equivalent to doing neither, so that $(N_2, \partial F') \cong (F' \times I, \partial F')$, which is clearly a product sutured manifold. This completes the proof of Theorem 1. □

Remark 2. Observe that this does not necessarily imply that L' is not fibered. It is possible that L' fibers with a different surface as a fiber. We combine our results with those of Kobayashi to address this question when the manifold is a rational homology 3-sphere in Section 4.

4. Characterization of band surgeries on fibered links

In this section we will characterize band surgeries. Throughout this section, L and L' are oriented links in S^3 related by a coherent band surgery along a band b. More precisely, b is an embedding $[0,1]\times[0,1] \to S^3$ such that $b^{-1}(L) = [0,1]\times\{0,1\}$, $b^{-1}(L') = \{0,1\} \times [0,1]$, and L and L' are the same as oriented sets except on $b([0,1] \times [0,1])$. For simplicity, we use the same symbol b to denote the image $b([0,1] \times [0,1])$. Since the numbers of components of L and L' differ by 1, their Euler characteristics will never be

equal. By [21, 12, 4], there exists a taut Seifert surface F for L or L', say L, such that F contains b, and so $\chi(L') > \chi(L)$.

Theorem 5 ([21, 12, 4])**.** *$\chi(L') > \chi(L)$ if and only if L has a taut Seifert surface F containing b.*

Suppose L is a fibered link. Then F is a fiber surface for L, and the band b is contained in F. Call $\alpha := b(\{\frac{1}{2}\} \times [0,1])$ the *spanning arc* of the band. The surface F', which is obtained by cutting F along α, can be regarded as a Seifert surface for L' by moving F' slightly along b. Note that α is fixed by the monodromy of F if and only if F' is split union of two fiber surfaces, *i.e.* L is a connected sum of a split link L'. Kobayashi characterized band surgeries in the case of $\chi(L') > \chi(L) + 1$ as follows. By Kobayashi [15] and Yamamoto [24], we have the following.

Theorem 6. *Suppose L is a fibered link. Then the following conditions are equivalent.*
(1) *$\chi(L') > \chi(L) + 1$.*
(2) *F' is a pre-fiber surface.*
(3) *There exists a disk D (of stallings twist) such that the intersection of D and F is a disjoint union of ∂D and α.*
(4) *α is clean and non-alternating but not fixed by the monodromy.*

See [15] for the definition of pre-fiber surfaces. Moreover Kobayashi showed the following.

Theorem 7. [15] *Suppose F is a fiber surface and F' is pre-fiber surface, then the band b is "type* F*" with respect to F'.*

See [16] for the definition of type F. He also characterized pre-fiber surfaces for fibered links in [15, Theorem 3] and for split links in [16, Theorem 3]. In particular, together with Theorem 6 and [15, Theorem 3], Theorem 7 gives a complete characterization of band surgeries between fibered links L and L' with $\chi(L') > \chi(L) + 1$.
Proof of Theorem 6.
(1)$\Rightarrow$(2) See Kobayashi [15, Theorem 2.1].
(2)$\Rightarrow$(1) If F' is a pre-fiber surface, by the definition of pre-fiber surfaces, then is compressible, and so $\chi(L') > \chi(F') = \chi(F) + 1 = \chi(L) + 1$.
(2)$\Rightarrow$(3) It follows from Theorem 7 and the definition of type F.
(3)$\Leftrightarrow$(4) See Yamamoto [24, Lemma 3.4].
(4)$\Rightarrow$(2) See Kobayashi [15, Proposition 4.5]. □

For the remaining case, we will characterize band surgeries between L and L' with $\chi(L') = \chi(L) + 1$. In this case, F is a fiber surface. By Theorem 1, then F' is a fiber surface for L' if and only if α is clean and alternating, or once-unlean and non-alternating. For the band b, we observe these conditions of α.

4.1. **Hopf banding and generalized Hopf banding.** First we show that if the spanning arc of a band surgery is a clean alternating arc, then the band surgery corresponds to a Hopf plumbing. If F is obtained by plumbing of a surface F'' and a Hopf annulus A, then F is obtained by attaching a band $\overline{A \setminus F''}$ to F'', and so we call F a *Hopf banding* of F'' along $\overline{A \setminus F''}$. Yamamoto [24, Proposition 2.3], Coward and Lackenby [3, Theorem 2.3] showed the following.

Theorem 8. *Suppose F is a fiber surface in S^3. Then F is a Hopf banding of F' if and only if α is clean and alternating.*

Next we introduce a "generalized Hopf banding" to correspond to a once-unclean non-alternating arc. Remark that a clean alternating arc α can be moved to be non-alternating by adding an unnecessary intersection point with $h(\alpha)$. Hence we can say that the band surgery for a once-unclean non-alternating arc is a generalization of Hopf banding.

Definition. Let ℓ be an arc in F' such that ℓ has a self intersection point and $\ell \cap \partial F' = \partial \ell$. Let b be a once-overlapped band over F' such that $b([0,1] \times \{\frac{1}{2}\})$ is parallel to ℓ, see Figure 6. The surface F is obtained by attaching b to F'. We call F a *generalized Hopf*

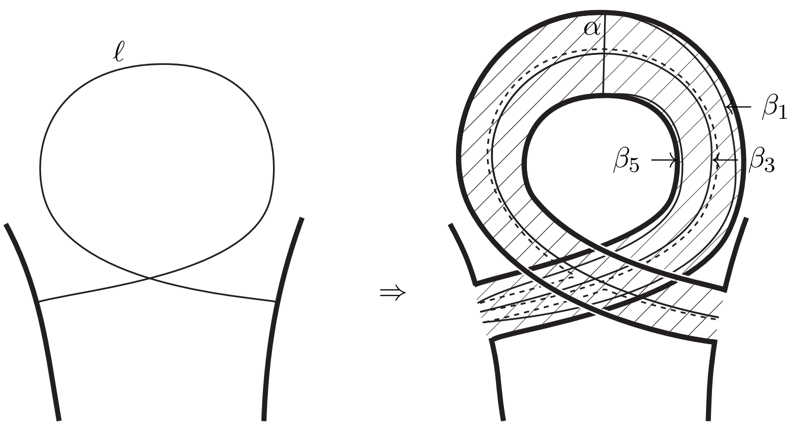


Figure 6

banding of F' along b.

Example. By generalized Hopf banding of a Hopf annulus, we can obtain two different 3-component fibered links (see Figure 7).

Note that, for each arc in F' having a self intersection point, we have two choices of generalized bandings according to the overlapped sides. Moreover, any Hopf banding is a generalized Hopf banding for ℓ whose self intersection point is removable by isotopies in F'. Then we have the following.

Theorem 9. *Suppose F is a fiber surface. Then F is not a Hopf banding but a generalized Hopf banding of F' if and only if α is once-unclean and non-alternating.*

Proof. Suppose F is a generalized Hopf banding of F' along a band b. Let $b' : [0,1] \times [0,1] \to F$ be a projection of $b : [0,1] \times [0,1] \to S^3$ into F', and put $I_i := [\frac{i}{5}, \frac{i+1}{5}]$ for $i \in \{0,1,2,3,4\}$. We may assume that $b'([0,1] \times \{\frac{1}{2}\}) = \ell$ and $b'|_{I_1 \times [0,1]}(\frac{s+1}{5}, t) = b'|_{I_3 \times [0,1]}(\frac{4-t}{5}, s)$ for $(s,t) \in [0,1] \times [0,1]$, and so the self intersection of ℓ is $b'(\frac{3}{10}, \frac{1}{2}) =$

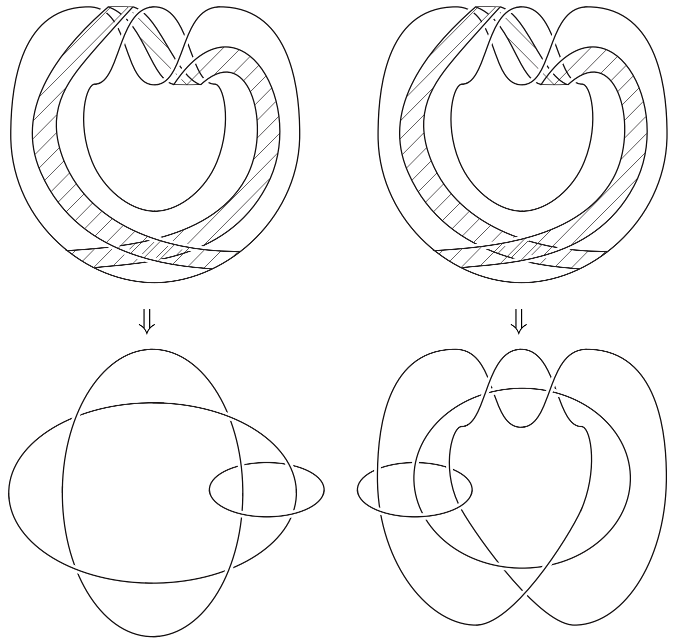


FIGURE 7. Generalized Hopf banding of Hopf annulus

$b'(\frac{7}{10}, \frac{1}{2})$. We also assume that $b(\frac{7}{10}, \frac{1}{2})$ is over $b(\frac{3}{10}, \frac{1}{2})$. Let $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ be arcs in F ($\beta_1, \beta_3, \beta_5 \subset b([0,1] \times [0,1])$ and $\beta_2, \beta_4, \subset F'$) defined by the following, see Figure 6.

$$\begin{aligned}
\beta_1 &:= \{b(s, \frac{1-2s}{3}) \mid 0 \leq s \leq \frac{1}{2}\} \\
\beta_2 &:= \{b'(s, \frac{1}{3}) \mid 0 \leq s \leq \frac{3}{10}\} \cup \{b'(s, \frac{1}{2}) \mid \frac{11}{15} \leq s \leq 1\} \\
\beta_3 &:= \{b(s, \frac{1}{2}) \mid 0 \leq s \leq 1\} \\
\beta_4 &:= \{b'(s, \frac{2}{3}) \mid 0 \leq s \leq \frac{3}{10}\} \cup \{b'(s, \frac{1}{2}) \mid 0 \leq s \leq \frac{2}{3}\} \\
\beta_5 &:= \{b(s, \frac{2+2s}{3}) \mid 0 \leq s \leq \frac{1}{2}\}
\end{aligned}$$

Put $\beta := \beta_1 \cup \beta_2 \cup \beta_3 \cup \beta_4 \cup \beta_5$. Then $h(\alpha)$ is isotopic to β in F, since β is a proper arc in F with $\partial\beta = \partial\alpha$ and $\alpha \cup \beta$ bounds a disk in the complement of F. The end points of β emanate to the same side of α and $int(\alpha) \cap int(\beta) = b(\frac{1}{2}, \frac{1}{2})$. Now b is not the band of a Hopf banding, and so α is not clean alternating by Theorem 8. Therefore α is once-unclean and non-alternating.

Suppose α is once-unclean and non-alternating. Put $\beta := h(\alpha)$. Then β is divided into five arcs $\beta_1, \beta_2, \beta_3, \beta_4, \beta_5$ by cutting along $b(\{0,1\} \times [0,1])$ so that β_i connects β_{i-1} and β_{i+1} for $i \in \{1,2,3,4,5\}$, $\beta_0 = \beta_6 = \alpha$. We may assume that $\beta_1, \beta_3, \beta_5$ are represented as above. Put $\ell' := \beta_2 \cup \{b(0,s) \mid \frac{1}{3} \leq s \leq \frac{2}{3}\} \cup \beta_4$. The arc ℓ' attaches to $\partial F'$ at $\{b(0,s) \mid \frac{1}{3} \leq s \leq \frac{2}{3}\}$. We have an arc ℓ with a single self intersection point by moving

ℓ' slightly into the interior of F'. Then F is a generalized Hopf banding of F' for ℓ, and not a Hopf banding by Theorem 8. $\square$

4.2. **Generalized Hopf banding for fiber surfaces.** It is well knowen that a Hopf banding is a fiber surface if and only if the original is fiber surface. In general a resulting surface of a Murasugi sum is a fiber surface if and only if the summands are both fiber surfaces [6, 7]. We have a similar result for generalized Hopf bandings.

Theorem 10. *Suppose F and F' are surfaces in S^3 such that F is a generalized Hopf banding of F'. Then F is a fiber surface if and only if F' is a fiber surface.*

Proof. One direction follows from Theorems 1, 8, and 9.

We will show that the complimentary sutured manifold $(S^3 \setminus n(F), \partial F)$ is trivial, and so F is a fiber surface. As in the proof of Theorem 9, $\alpha \cup \beta$ bounds a disk in the complement of F. From the disk, we have the product disk D for $(S^3 \setminus n(F), \partial F)$. Note that $n(F)$ is obtained from $n(F')$ by attaching a 1-handle $n(b)$. Since $int(\alpha)$ intersects $int(\beta)$ at a point, and β emanates away from α in the same direction at both endpoints of α, ∂D intersects a co-core of the 1-handle at a point, and so D cancels the 1-handle. Then $(S^3 \setminus n(F), \partial F)$ is decomposed into $(S^3 \setminus n(F'), \partial F')$ by D.

$$(S^3 \setminus n(F), \partial F) \overset{D}{\rightsquigarrow} (S^3 \setminus n(F'), \partial F')$$

Since F' is a fiber surface, $(S^3 \setminus n(F'), \partial F')$ is a trivial sutured manifold, and so $(S^3 \setminus n(F), \partial F)$ is also trivial. Hence F is a fiber surface. $\square$

By Theorems 1, 8, 9, and 10, we have the following:

Theorem 11. (1) *Suppose F is a fiber surface in S^3 and b is a band in F such that $b \cap \partial F = b([0,1] \times \{0,1\})$. Put $F' := \overline{F \setminus b}$. Then F' is a fiber surface if and only if F is a generalized Hopf banding of F' along b.*

(2) *Suppose F' is a fiber surface in S^3 and b is a band attaching F', i.e. $b \cap F' = b(\{0,1\} \times [0,1]) \subset \partial F'$. Put $F := F' \cup b$. Then F is a fiber surface if and only if F is a generalized Hopf banding of F' along b.*

Theorem 5 implies that any coherent band surgery on links can be regarded as an operation of cutting a taut Seifert surface along the band. Then as a translation of Theorem 11, we have proven Theorem 2.

Theorem 2. *Suppose L' is obtained from L by a coherent band surgery and $\chi(L') = \chi(L) + 1$.*

(1) *Suppose L is a fibered link. Then L' is a fibered link if and only if the fiber F for L is a generalized Hopf banding of a Seifert surface F' for L' along b.*

(2) *Suppose L' is a fibered link. Then L is a fibered link if and only if a Seifert surface F for L is a generalized Hopf banding of the fiber F' for L' along b.*

It is well known that any automorphism of a surface can be represented by a composition of Dehn twists. Let F be a fiber surface with monodromy h. Honda, Kazez, and Matic [13] showed the following:

Lemma 3. [13, Lemma 2.5] *Suppose h is a composition of right hand Dehn twists along circles in F. Then h is right-veering, i.e. any arc α in F is alternating, otherwise $h(\alpha)$*

is isotopic to α in F (α is non-alternating and clean). In other words, $i_\partial(\alpha, h(\alpha)) = 1$ if $h(\alpha)$ is not isotopic to α.

Remark that $h(\alpha)$ is isotopic to α if and only if there exists a 2-sphere S such that $S \cap F = \alpha$. If we assume additionally that ∂F is prime, then any essential arc in F is alternating. We will discuss the case where ∂F is composite in Subsection 4.4.

Suppose a fiber surface F with monodromy h is obtained by plumbing of two surfaces F_1 and F_2, where F_1 is a Hopf annulus with left hand twist. Let C be a core circle of F_1. We denote by t_C the right hand Dehn twist along C. Then $(t_C^{-1} \circ h)|_{F_2}$ is isotopic to the monodromy for F_2. Hence if F is obtained from a disk in S^3 by successively plumbing Hopf annuli with right hand twist, then h is a composition of right hand Dehn twists. By Theorem 2 (1) and Lemma 3, we have the following:

Corollary 1. *Let L be an oriented link with fiber F such that F is obtained from a disk by successively plumbing Hopf annuli with right hand twist (or by successively plumbing Hopf annuli with left hand twist). Suppose $\chi(L') = \chi(L) + 1$. Then L' is a fibered link if and only if F is a Hopf banding of a Seifert surface F' for L' along b.*

4.3. **Band surgeries on $(2,p)$-torus link.** Let D_1 and D_2 be disjoint disks in a plane. Let $b_1, \ldots, b_p$ be pairwise disjoint bands with left hand half twist connecting the two disks. Put $F := D_1 \cup D_2 \cup b_1 \cup \cdots \cup b_p$. Then F is a fiber surface for the $(2,p)$-torus link $T(2,p)$ (with parallel orientation if p is even) (see Figure 8). Let b be a band in F,

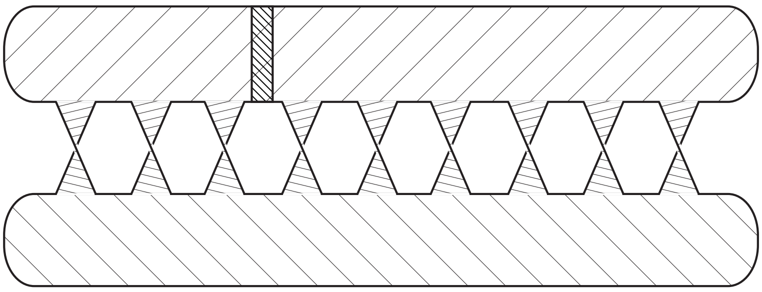


FIGURE 8. Fiber surface for $(2,9)$-torus knot

and put $F' := \overline{(F \setminus b)}$, $L' := \partial F'$. Since F is obtained from $D_1 \cup D_2 \cup b_1 \cup \cdots \cup b_{p-1}$ by plumbing Hopf annuli with left hand twist along b_{p-1} (or its core arc), F is obtained from a disk $D_1 \cup D_2 \cup b_1$ by successively plumbing $(p-1)$ Hopf annuli with left hand twist. Then, by Corollary 1, L' is fibered if and only if F is a Hopf banding of F' along b.

Corollary 2. *Suppose L' is obtained from $L = T(2,p)$ by a coherent band surgery along b, where $p \geq 2$, and $\chi(L') > \chi(L)$. Then L' is fibered if and only if the band b can be moved into D_1 (and also D_2) so that $F \setminus b$ is connected. In particular, if we assume L' is a prime (resp. composite) fibered link, then L' is $T(2,p-1)$ (resp. a connected sum $T(2,p_1)\#T(2,p_2)$ of $T(2,p_1)$ and $T(2,p_2)$, where p_1 and p_2 are positive integers with $p_1, p_2 > 1$ and $p_1 + p_2 = p$). Moreover, the band is unique up to isotopy fixing L as a set if $L' = T(2,p-1)$ or $T(2,p_1)\#T(2,p_2)$ and either p_1 or p_2 is odd, and there are two*

bands up to isotopy fixing L as a set if both p_1 and p_2 are even (L' is a 3-component link), but they are the same up to homeomorphism.

Remark 3. By Murasugi [17], $|\sigma(L) - \sigma(L')| \leq 1$ for two links L, L' which are related by a coherent band surgery, where σ means the signature. Since $\chi(T(2,p)) = 2 - p$ and $\sigma(T(2,p)) = 1 - p$, $\chi(L) + 1 = \sigma(L) \geq \sigma(L') - 1 \geq \chi(L')$ if $L = T(2,p)$. Then the assumption $\chi(L') > \chi(L)$ in Corollary 2 becomes $\chi(L') = \chi(L) + 1$. Remark that we can regard $T(2, p-1)$ as $T(2,p_1)\#T(2,p_2)$ for $p_1 = p - 1$ and $p_2 = 1$ since $T(2,1)$ is a trivial.

Proof. Suppose that b is contained in F, disjoint from $b_1, \ldots, b_p$, and does not separate F. We will prove that F' is fibered, and that the band is unique up to the operations mentioned. Say b is contained in D_1, and b splits D_1 into two disks with p_i bands of $b_1, \ldots, b_p$ ($i = 1, 2$), where p_1 and p_2 are positive integers with $p_1 + p_2 = p$. Then L' is a connected sum $T(2,p_1)\#T(2,p_2)$ of $T(2,p_1)$ and $T(2,p_2)$ which is a fibered link. Moreover two such bands in F are related by the monodromy and sliding along ∂F if the two bands are attached to the same component of L. This implies that the band is unique up to isotopies fixing L as a set if either p_1 or p_2 is odd. If the two bands are attached to different components of L, they are related by the monodromy, sliding along ∂F, and an involution. Here we can take a rotation about the horizontal axis in Figure 8 as the involution so that D_1 is mapped to D_2, D_2 is mapped to D_1, and b_i is mapped to itself. This implies that the two bands are the same up to homeomorphism.

Conversely, let α be a clean and alternating arc in F. We will show that α can be moved into D_1 or D_2 so that α is disjoint from $b_1, \ldots, b_p$. This will show that any band producing a fibered link L' can be moved into D_1 or D_2 by Corollary 1 and Theorem 8. We arrange the bands $b_1, \ldots, b_p$ along an orientation of ∂D_1 (or ∂D_2). Let $\widehat{h}$ be an automorphism of F such that $\widehat{h}(D_1) = D_2, \widehat{h}(D_2) = D_1$, and $\widehat{h}(b_i) = b_{i+1} (\mathrm{mod}\ p)$. In other words, $\widehat{h}$ is obtained from the monodromy h of F by sliding to the left hand side along ∂F so that $\widehat{h}(D_1) = D_2, \widehat{h}(D_2) = D_1$. Since α and $h(\alpha)$ intersect only at their endpoints with positive signs, α is disjoint from $\widehat{h}(\alpha)$. We may assume that α minimizes intersections with $int(b_1 \cup \cdots \cup b_p)$, and $\partial\alpha$ consists of two points of $L \cap (D_1 \cup D_2) \cap (b_1 \cup \cdots \cup b_p)$. For a contradiction, suppose α intersects $int(b_1 \cup \cdots \cup b_p)$. Then α is divided into arcs by cutting F along $b_1 \cup \cdots \cup b_p$. Let α_1, α_2 be successive such arcs in D_1, D_2 respectively, and define the following (see Figure 9).

(1) $\partial\alpha_1 = \{x, y\}$, where x is a point in $\partial D_1 \cap \partial b_i$ and y is a point in $\partial D_1 \cap \partial b_j$.

(2) $\partial\alpha_2 = \{z, w\}$, where z is a point in $\partial D_2 \cap \partial b_j$ and w is a point in $\partial D_2 \cap \partial b_k$.

(3) A component of $\alpha \cap b_j$ connects y and z in b_j.

Put $\beta_1 := \widehat{h}(\alpha_1), \beta_2 := \widehat{h}(\alpha_2)$, $x' := \widehat{h}(x), y' := \widehat{h}(y), z' := \widehat{h}(z), w' := \widehat{h}(w)$. Then x', y', z', w' are points in $\partial D_2 \cap \partial b_{i+1}, \partial D_2 \cap \partial b_{j+1}, \partial D_1 \cap \partial b_{j+1}, \partial D_1 \cap \partial b_{k+1}$ respectively.

First we show that $j - i \equiv \pm 1 \pmod{p}$ or $k - j \equiv \pm 1 \pmod{p}$. Suppose $k - j \not\equiv \pm 1$ $(\mathrm{mod}\ p)$. Let D_1' (resp., D_2') be a disk cut off from D_1 by β_2 (resp., D_2 by α_2) which bands $b_{k+2}, \ldots, b_j \pmod{p}$, a half of b_{k+1} and a half of b_{j+1} are attaching (resp., bands $b_{j+1}, \ldots, b_{i-1} \pmod{p}$ a half of b_j and a half of b_i are attaching). Since α_1 is disjoint from β_2 in D_1, two points x and y both lie in D_1', and so $i \equiv k+1, k+2, \ldots, j-1$, or $j + 1 \pmod{p}$. On the other hand, since α_2 is disjoint from β_1 in D_2, two points x' and

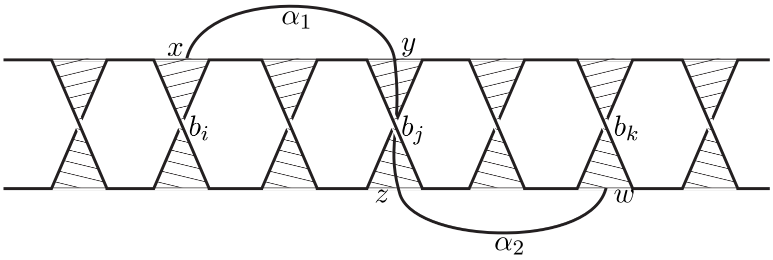


FIGURE 9

y' both lie in D_2', and so $i+1 = j, j+2, j+3, \ldots, k-1$, or k (mod p). Then $j - i \equiv \pm 1$ (mod p).

Next we show that if α_1 is outermost in D_1 and $j-i \equiv 1$ (mod p), then α_2 is outermost in D_2 and $k - j \equiv 1$ (mod p). Similarly, if α_2 is outermost in D_2 and $k - j \equiv -1$ (mod p), then α_1 is outermost in D_1 and $j - i \equiv -1$ (mod p). Suppose that α_1 is outermost in D_1 and $j = i+1$ ($j = 1$ if $i = p$) . Then β_1 connects a point x' in $\partial D_2 \cap \partial b_{i+1}$ and a point y' in $\partial D_2 \cap \partial b_{i+2}$. Recall that z is a point in $\partial D_2 \cap \partial b_{i+1}$. Since α_1 is outermost in D_1, z lies in outer side of β_1, and so α_2 is parallel to β_1 and outermost in D_2.

Finally we show that this results in a contradiction. Suppose α has a sub-arc α' which is outermost in D_1 or D_2 and connecting two adjacent bands. By continuing the same argument above, we may assume that the end subarc α_2 (or α_1) of α is outermost in D_1 or D_2 and $k - j \equiv 1$ (mod p) (or $j - i \equiv -1$ (mod p)). Then α has a removable intersection with b_j. In the case where $\widehat{h}(\alpha)$ has a subarc which is outermost in D_1 or D_2 and connecting two adjacent bands, by the same argument, $\widehat{h}(\alpha)$ (and α does so) has a removable intersection with $b_1, \ldots, b_p$. This contradicts the assumption that α minimizes intersections with $b_1, \ldots, b_p$. $\square$

4.4. **Band surgeries on composite fibered links.** We say that a fiber surface is *prime* (resp., *composite*) if the boundary is a prime link (resp., a composite link). Suppose F is a composite fiber suface. There exists a 2-sphere S intersecting F in an arc, such that neither surface cut off from F by the arc is a disk. The resulting surfaces are both fiber surfaces for the summand links. In general, there exist pairwise disjoint 2-spheres $S_1, \ldots, S_m$ such that $\delta_i := S_i \cap F$ is an arc for each $i \in \{1, \ldots, m\}$, and each component of the surface obtained from F by cutting along $\delta_1, \ldots, \delta_n$ is a prime fiber surface. We call a set $\{\delta_1, \ldots, \delta_m\}$ of such arcs a *full prime decomposing system* for F. Remark that if $m = 1$, a full prime decomposing system (an arc in this case) is unique up to isotopy in F. On the other hand, there exist several decomposing systems if $m \geq 2$, but the sets of surfaces obtained from F by cutting along decomposing systems are always the same.

Suppose a fiber surface F is divided into prime fiber surfaces $F_1, \ldots, F_{m+1}$, and a properly embedded arc α in F is divided minimizingly into subarcs $\alpha_1, \ldots, \alpha_n$ successively by a full prime decomposing system $\{\delta_1, \ldots, \delta_m\}$, where α_i is a properly embedded arc in F_{j_i} for each $i \in \{1, \ldots, n\}$ and $\{p_i\} = \alpha_i \cap \alpha_{i+1} \subset \partial\alpha_i, \partial\alpha_{i+1}$ for each $i \in \{1, \ldots, n-1\}$. Let $s_i, t_{i+1} = \pm 1$ be the signs at p_i for a pair $(\alpha_i, h_{j_i}(\alpha_i))$ in F_{j_i} and

for a pair $(\alpha_{i+1}, h_{j_{i+1}}(\alpha_{i+1}))$ in $F_{j_{i+1}}$ respectively. Remark that $i_\partial(\alpha_i, h_{j_i}(\alpha)) = \frac{t_i+s_i}{2}$ for each $i \in \{2, \ldots, n-1\}$, see [11]. Then we have the following.

Lemma 4.

$$\rho(\alpha) = \sum_{i=1}^{n} \rho(\alpha_i) + \frac{1}{2}\sum_{i=1}^{n-1} |s_i + t_{i+1}|$$

Here if $h_{j_i}(\alpha_i)$ is isotopic to α_i in F_{j_i}, $(t_i, s_i) = (1, -1)$ or $(-1, 1)$ which minimizes $\sum_{i=1}^{n-1} |s_i + t_{i+1}|$.

Proof. First we will show that $\rho(\alpha) \leq \sum_{i=1}^{n} \rho(\alpha_i) + \frac{1}{2}\sum_{i=1}^{n-1} |s_i + t_{i+1}|$. We can take the monodromies h, $h_1, \ldots, h_{m+1}$ of F, $F_1, \ldots, F_{m+1}$ respectively, and $\alpha_1, \ldots, \alpha_n$ so that $h|_{F_j} = h_j$ and $|int(\alpha_i) \cap int(h(\alpha_i))| = \rho(\alpha_i)$. By moving h slightly at p_i if $s_i + t_{i+1} = 0$, then, we have $\rho(\alpha) \leq |int(\alpha) \cap int(h(\alpha))| = \sum_{i=1}^{n} \rho(\alpha_i) + \frac{1}{2}\sum_{i=1}^{n-1} |s_i + t_{i+1}|$.

Next we will show that $\rho(\alpha) \geq \sum_{i=1}^{n} \rho(\alpha_i) + \frac{1}{2}\sum_{i=1}^{n-1} |s_i + t_{i+1}|$. Put $\beta := h(\alpha)$ so that $|int(\alpha) \cap int(\beta)| = \rho(\alpha)$. Then there exists a disk D, possibly with self intersection in the boundary, such that $D \cap F = \partial D = \alpha \cup \beta$. We observe the intersection of D and pairwise disjoint spheres $S_1, \ldots, S_m$, where $S_i \cap F = \delta_i$ for each $i \in \{1, \ldots, m\}$. By cut and past argument, we may assume that the intersection $D \cap (S_1 \cup \cdots \cup S_m)$ consists of arcs. Let D' be an outermost disk cut off from D by $D \cap S_i$ for some $i \in \{1, \ldots, m\}$. Suppose $\partial D' \cap \partial D \subset \alpha$ or $\partial D' \cap \partial D \subset \beta$. Since F is incompressible, $\partial D' \cap \partial D$ is isotopic in F to a sub-arc of δ_i joining the end points of the arc of $D \cap S_i$. Hence such arc of the intersection $D \cap S_i$ is removable keeping $|int(\alpha) \cap int(\beta)|$ constant. After removing such intersections, D is divided into disks $D_1, \ldots, D_n$ by $D \cap (S_1 \cup \cdots \cup S_m)$, where ∂D_i consists of α_i, a sub-arc β_i of β, and two parallel arcs (resp., a single arc) of $D \cap (S_1 \cup \cdots \cup S_m)$ for each $i \in \{2, \ldots, n-1\}$ (resp., $i \in \{1, n\}$) (see Figure 10). This implies that $h_{j_i}(\alpha_i)$ is isotopic to β_i. By sliding β_i along ∂F_{j_i} in F_{j_i} so that the end points of β_i coincide with those of α_i, we have $\rho(\alpha) = |int(\alpha) \cap int(\beta)| = \sum_{i=1}^{n} |int(\alpha_i) \cap int(\beta_i)| + \frac{1}{2}\sum_{i=1}^{n-1} |s_i + t_{i+1}| \geq \sum_{i=1}^{n} \rho(\alpha_i) + \frac{1}{2}\sum_{i=1}^{n-1} |s_i + t_{i+1}|$. $\square$

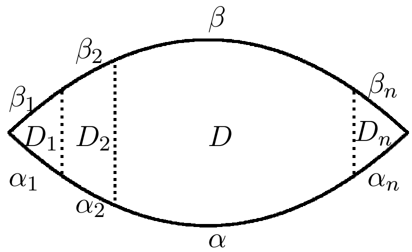


FIGURE 10

By Lemma 3 and Lemma 4, we have the following.

Theorem 12. *Suppose that a fiber surface F is divided into fiber surfaces $F_1, \ldots, F_{m+1}$, and a properly embedded arc α in F is divided minimizingly into subarcs $\alpha_1, \ldots, \alpha_n$ successively by a full prime decomposing system $\{\delta_1, \ldots, \delta_m\}$, where the monodromy h_j of F_j is a composition of right hand Dehn twists or left hand Dehn twists according to*

whether $\varepsilon_j = +$ or $-$, and α_i is a properly embedded arc in F_{j_i}. Then the following holds:

(1) *The arc α is clean and alternating if and only if the set $\{1, \dots, n\}$ is partitioned into two sets A and B, with A consisting of an odd number of elements, so that α_i is clean and alternating in F_{j_i} and ε_{j_i} appears as $+$ and $-$ alternatively in ascending order for $i \in A$, and α_i is parallel to the boundary ∂F_{j_i} in F_{j_i} for any $i \in B$.*

(2) *The arc α is once-unclean and non-alternating if and only if either:*

   (2-1) *the set $\{1, \dots, n\}$ is partitioned into two sets A and B, with A consisting of an even number of elements, so that α_i is clean and alternating in F_{j_i} except for one once-unclean alternating arc and ε_{j_i} appears as $+$ and $-$ alternatively in ascending order for $i \in A$, and α_i is parallel to the boundary ∂F_{j_i} in F_{j_i} for any $i \in B$, or*

   (2-2) *the set $\{1, \dots, n\}$ is partitioned into two sets A and B, with A consisting of an odd number of elements, so that α_i is clean and alternating in F_{j_i} and ε_{j_i} appears as $+$ and $-$ alternatively except one successive pair in ascending order for $i \in A$, and α_i is parallel to the boundary ∂F_{j_i} in F_{j_i} for any $i \in B$.*

Proof. By Lemma 3, α_i is non-alternating ($t_i + s_i = 0$) if and only if $h_{j_i}(\alpha_{j_i})$ is isotopic to α_i in F_{j_i}, and if α_i is alternating ($t_i + s_i \neq 0$) then $t_i = s_i = \varepsilon_{j_i}$. Since F_{j_i} is prime, $h_{j_i}(\alpha_i)$ is isotopic to α_i in F_{j_i} if and only if α_i is parallel to the boundary ∂F_{j_i} in F_{j_i}. Partition the set $\{1, \dots, n\}$ into A and B so that α_i is alternating if $i \in A$, and parallel to the boundary ∂F_{j_i} in F_{j_i} if $i \in B$. Suppose that $i \in A$ and $i+1 \in B$ (resp., $i-1 \in B$), then $\rho(\alpha_i) = 0$, and (t_i, s_i) can be taken as $(\varepsilon_{i+1}, -\varepsilon_{i+1})$ (resp., $(-\varepsilon_{i-1}, \varepsilon_{i-1})$) so that s_i and t_{i+1} (resp., s_{i-1} and t_i) are cancelled. Hence we can ignore the elements of B when we calculate the the number $\sum_{i=1}^{n} \rho(\alpha_i) + \frac{1}{2}\sum_{i=1}^{n-1} |s_i + t_{i+1}|$. Remark that $i_\partial(\alpha, h(\alpha)) = \varepsilon_{j_i} + \varepsilon_{j_{i'}}$, where i and i' are the first and the last elements of A.

(1) By the definition, the arc α is clean and alternating if and only if $\rho(\alpha) = 0$ and $i_\partial(\alpha, h(\alpha)) = \pm 1$. By Lemma 4, $\rho(\alpha) = 0$ if and only if α_i is clean for each $i \in A$, and $\varepsilon_{j_i} + \varepsilon_{j_{i'}} = 0$ for each pair of successive integers i and i' in A. Then (1) of Theorem 12 holds.

(2) By the definition, the arc α is once-unclean and non-alternating if and only if $\rho(\alpha) = 1$ and $i_\partial(\alpha, h(\alpha)) = 0$. By Lemma 4, $\rho(\alpha) = 1$ if and only if either: (2-1) α_i is clean for $i \in A$ except one once-unclean, and $\varepsilon_{j_i} + \varepsilon_{j_{i'}} = 0$ for a pair of successive integers i and i' in A, or (2-2) α_i is clean for $i \in A$. and $\varepsilon_{j_i} + \varepsilon_{j'_i} = 0$ for a pair of successive integers i and i' in A except for one pair. Then (2) of Theorem 12 holds. □

The following corollary is derived from Theorem 12 by considering the case when $\varepsilon_1 = \cdots = \varepsilon_{m+1} = +$ or $\varepsilon_1 = \cdots = \varepsilon_{m+1} = -$.

Corollary 3. *Suppose a fiber surface F is composite, has monodromy which is a composition of right hand Dehn twists or left hand Dehn twists, and that an arc α in F is clean and alternating. Then there exists a full prime decomposing system $\{\delta_1, \dots, \delta_m\}$ such that α is disjoint from $\delta_1 \cup \cdots \cup \delta_m$.*

Proof. Let $\{\delta_1, \dots, \delta_m\}$ be a full prime decomposing system for F, so that F is divided into fiber surfaces $F_1, \dots, F_{m+1}$ by $\{\delta_1, \dots, \delta_m\}$. Suppose that a clean alternating arc α in F is divided minimizingly into subarcs $\alpha_1, \dots, \alpha_n$ ($n \geq 2$) successively by $\{\delta_1, \dots, \delta_m\}$,

i.e. $|\alpha \cap (\delta_1 \cup \cdots \cup \delta_m)| = n-1$. Then the monodromy h_j of F_j is a composition of right hand Dehn twists or left hand Dehn twists according to whether that of F is a composition of right hand Dehn twists or left hand Dehn twists. By Theorem 12 (1), there exists $k \in \{1, \ldots, n\}$ such that α_k is clean and alternating, and any other arc α_i ($i \in \{1, \ldots, n\} - \{k\}$) is parallel to ∂F_{j_i} in F_{j_i} (*i.e.* the set A in Theorem 12 (1) must be a singleton set $\{k\}$ in this case). Without loss of generality, we may assume that $k \neq n$, α_{n-1} and α_n are arcs in F_m and F_{m+1} respectively, and $F_m \cap F_{m+1} = \delta_m$. Since α_n is parallel to ∂F_{m+1} in F_{m+1}, α_n divides F_{m+1} into a disk D and a surface F'_{m+1} which is homeomorphic to F_{m+1}, and divides δ_m into a and b, where $a \subset \partial F'_{m+1}$ and $b \subset \partial D$. Let δ'_m be an arc obtained from $\alpha_n \cup a$ by pushing slightly into the interor of F'_{m+1}. Then the set $\{\delta_1, \ldots, \delta_{m-1}, \delta'_m\}$ is a new full prime decomposing system which divides F into $F_1, \ldots, F_{m-1}, F'_m, F'_{m+1}$, where $F'_m = F_m \cup D$, and $|\alpha \cap (\delta_1 \cup \cdots \cup \delta_{m-1} \cup \delta'_m)| < n-1$. By continuing such operations, we obtain a full prime decomposing system for F which is disjoint from α. □

Hence, in Corollary 1, if we assume that L is composite, we can take decomposing spheres for L so that the band of a Hopf banding is disjoint from the decomposing spheres. Then we have the following from Corollary 2.

Corollary 4. *Suppose L' is obtained from $L = T(2,p)\#T(2,q)$ by a coherent band surgery along b and $\chi(L') > \chi(L)$, where $p, q > 1$. If L' is fibered, then L' is a connected sum $T(2,p_1)\#T(2,p_2)\#T(2,q)$ or $T(2,p)\#T(2,q_1)\#T(2,q_2)$, where p_1, p_2, q_1, q_2 are positive integers with $p_1 + p_2 = p$, $q_1 + q_2 = q$. Moreover, for each $L' = T(2,p_1)\#T(2,p_2)\#T(2,q)$ or $T(2,p)\#T(2,q_1)\#T(2,q_2)$, the band is unique up to homeomorphisms.*

5. Characterization of crossing changes on fibered links

In this section we will characterize crossing changes between fibered links. Throughout this section, L and L' are oriented links in S^3 related by a crossing change. More precisely, there exists a disk D in S^3 such that L intersects D in two points of opposite orientations, and L' is the image of L after ± 1 Dehn surgery along $c = \partial D$. Scharlemann and Thompson [21] showed that there exists a taut Seifert surface F for L or L', say L, such that F is disjoint from c but intersects D in an arc.

Theorem 13 ([21])**.** *$\chi(L') \geq \chi(L)$ if and only if L has a taut Seifert surface F such that F is disjoint from c but intersects D in an arc. Moreover, $\chi(L') > \chi(L)$ if and only if F is a plumbing of a Hopf annulus A and a surface F'' which is disjoint from D, and the result A' of A after the twist is compressible.*

Suppose L is a fibered link. Let α be the arc $D \cap F$ in Theorem 13. Then we say that performing a ± 1-Dehn surgery along c is a ± 1-*twist along* α. Here an ε-twist is right- or left-handed according to whether $\varepsilon = 1$ or -1, respectively. By Theorem 8, we can restate the last part of Theorem 13 as follows.

Theorem 14. *Suppose L is a fibered link with fiber F, L' is obtained from L by the ε-twist along α ($\varepsilon = \pm 1$), where α is an arc in F. Then $\chi(L') > \chi(L)$ if and only if α is clean and alternating with $i_\partial(\alpha) = -\varepsilon$.*

Moreover, Kobayashi characterized crossing changes in the case of $\chi(L') > \chi(L)$ as follows. The surface F'' in Theorem 13 is also a fiber surface with $\chi(L') > \chi(L'')$, where $L'' = \partial F''$. By regarding a band b and the link L' as $b = F'' \cap A$ and $L' = \partial\overline{(F'' - A)}$, the characterization of band surgeries in Theorem 6 and Theorem 7 gives a characterization of crossing changes between a fibered link L and a link L' with $\chi(L) < \chi(L')$. Together with the characterization of pre-fiber surfaces for fibered links, moreover, it gives a complete characterization of crossing changes between fibered links L and L' with $\chi(L) < \chi(L')$. For the remaining case, we will characterize band surgeries between fibered links L and L' with $\chi(L) = \chi(L')$.

We remark that Stallings proved if F is a fiber surface, and the loop c is isotopic into F so that the framing on c induced by F agrees with that of D, then the image of F after ± 1-Dehn surgery along c is a new fiber surface for the resulting link [22]. This came to be known as a *Stallings twist*. Yamamoto proved that twisting along an arc is a Stallings twist if and only if the arc α is clean and non-alternating [24], see Theorem 6. We generalize this and characterize exactly when twisting along an arc results in a new fiber surface.

Theorem 15. *Suppose F is a fiber surface with monodromy h, and α is a properly embedded arc in F. Let F' be the resulting surface of an ε-twist along α. Then F' is a fiber surface if and only if $i_{total}(\alpha) = 0$ (i.e., α is clean and non-alternating) or α is once-unclean and alternating with $i_\partial(\alpha) = -\varepsilon$.*

Proof. Plumb a Hopf band along an arc parallel to the boundary of F, with endpoints on either side of α, so that $p \in \partial\alpha$ is in the trivial sub-disk cut off by this arc. The result is a new fibered link, together with its fiber, F''. Observe that the monodromy of F'' differs from that of F by exactly a Dehn twist along the core of the newly plumbed on Hopf band, right- or left-handed depending on the twist of the Hopf band.

By Theorem 1, the result of cutting F'' along α is a fiber if and only α is clean, alternating, or once-unclean, non-alternating in F''. The arc α will be clean, alternating in F'' exactly when α is clean, non-alternating in F and the sign of the Hopf band disagrees with the sign of $i(\alpha, h(\alpha))$ at p in F. The arc α will be once-unclean, non-alternating in F'' exactly when either α is clean, non-alternating in F and the sign of the Hopf band agrees the sign of $i(\alpha, h(\alpha))$ at p in F, or when α is once-unclean, alternating in F, and the sign of the Hopf band disagrees with the sign of $i(\alpha, h(\alpha))$ at p in F. □

By Theorem 13, we have Theorem 3 as a translation of Theorem 15.

Theorem 3. *Suppose a link L' is obtained from a fibered link L with fiber F by a crossing change, and $\chi(L') = \chi(L)$. Then L' is a fibered link if and only if the crossing change is a Stallings twist or ε-twist along an arc α in F, where α is once-unclean and alternating with $i_\partial(\alpha) = -\varepsilon$.*

References

[1] Dorothy Buck, Kai Ishihara, Matt Rathbun, and Koya Shimokawa. *in preparation*.

[2] Gerhard Burde and Heiner Zieschang. Neuwirthsche knoten und flächenabbildungen. *Abh. Math. Sem. Univ. Hamburg*, 31:239–246, 1967.

[3] Alexander Coward and Marc Lackenby. Unknotting genus one knots. *Comment. Math. Helv.*, 86:383–399, 2011.

[4] Arnaud Deruelle, Katsura Miyazaki, and Kimihiko Motegi. Networking Seifert surgeries on knots III. *preprint*, 2011.

[5] David Gabai. Foliations and the topology of 3-manifolds. *J. Differential Geom.*, 18(3):445–503, 1983.

[6] David Gabai. The Murasugi sum is a natural geometric operation. *Contemp. Math.*, 20:131–144, 1983.

[7] David Gabai. Detecting fibred links in S^3. *Comment. Math. Helv.*, 61(4):519–555, 1986.

[8] Emmanuel Giroux. Convexité en topologie de contact. *Comment. Math. Helv.*, 66:637–677, 1991.

[9] Emmanuel Giroux and Noah Goodman. On the stable equivalence of open books in three-manifolds. *Geom. Topol.*, 10:97–114 (electronic), 2006.

[10] Hiroshi Goda. Heegaard splitting for sutured manifolds and Murasugi sum. *Osaka J. Math.*, 29(1):21–40, 1992.

[11] Noah Goodman. Overtwisted open books from sobering arcs. *Algebr. Geom. Topol.*, 5:1173–1195 (electronic), 2005.

[12] Mikami Hirasawa and Koya Shimokawa. Dehn surgeries on strongly invertible knots which yield lens spaces. *Proc. Amer. Math. Soc.*, 128(11):3445–3451, 2000.

[13] Ko Honda, William H. Kazez, and Gordana Matić. Right-veering diffeomorphisms of compact surfaces with boundary. *Invent. math.*, 169:427–449, 2007.

[14] Rob Kirby ed. Problems in low-dimensional topology. In *Geometric topology (Athens, GA, 1993)*, volume 2 of *AMS/IP Stud. Adv. Math.*, pages 35–473. Amer. Math. Soc., Providence, RI, 1997.

[15] Tsuyoshi Kobayashi. Fibered links and unknotting operations. *Osaka J. Math.*, 26(4):699–742, 1989.

[16] Tsuyoshi Kobayashi. Fibered links which are band connected sum of two links. *Knots 90*, pages 9–24, 1992.

[17] Kunio Murasugi. On a certain numerical invariant of link types. *Trans. Amer. Math. Soc.*, 117:387–422, 1965.

[18] Yi Ni. Knot Floer homology detects fibred knots. *Invent. Math.*, 170(3):577–608, 2007.

[19] Peter Ozsváth and Zoltán Szabó. On knot Floer homology and lens space surgeries. *Topology*, 44(6):1281–1300, 2005.

[20] Martin Scharlemann. Sutured manifolds and generalized Thurston norms. *J. Differential Geom.*, 29(3):557–614, 1989.

[21] Martin Scharlemann and Abigail Thompson. Link genus and the Conway moves. *Comment. Math. Helv.*, 64(4):527–535, 1989.

[22] John R. Stallings. Constructions of fibred knots and links. In *Algebraic and geometric topology (Proc. Sympos. Pure Math., Stanford Univ., Stanford, Calif., 1976), Part 2*, Proc. Sympos. Pure Math., XXXII, pages 55–60. Amer. Math. Soc., Providence, R.I., 1978.

[23] William P. Thurston. On the geometry and dynamics of diffeomorphisms of surfaces. *Bull. Amer. Math. Soc. (N.S.)*, 19(2):417–431, 1988.

[24] Ryosuke Yamamoto. Stallings twists which can be realized by plumbing and deplumbing Hopf bands. *J. Knot Theory Ramifications*, 12(6):867–876, 2003.

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